

Singular Accelerated Evolution in massive $F(R)$ bigravity

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The possibility to have singular accelerated evolution in the context of $F(R)$ bimetric gravity is investigated. Particularly, we study two singular models of cosmological evolution, one of which is a singular modified version of the Starobinsky R^2 inflation model. As we demonstrate, for both models in some cases, the slow-roll parameters become singular at the Type IV singularity, a fact that we interpret as a dynamical instability of the theory under study. This dynamical instability may be an indicator of graceful exit from inflation and we thoroughly discuss this scenario and the interpretation of the singular slow-roll parameters. Furthermore, it is demonstrated that for some versions of $F(R)$ bigravity, singular inflation is realized in consistent way so that inflationary indices are compatible with Planck data. Moreover, we study the late-time behavior of the two singular models and we show that the unified description of early and late-time acceleration can be achieved in the context of bimetric $F(R)$ gravity.

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I. INTRODUCTION

Since the striking discovery of late-time acceleration in the late 90's [1], many theoretical descriptions have been proposed to model this late-time acceleration. This late-time acceleration contributes at almost 76% to the total energy density of the Universe, and it is known as dark energy. Modified theories of gravity provide a successful and self-consistent description of dark energy (see reviews [2]). On the same time, modified gravity may also successfully describe the early-time acceleration, i.e. inflationary stage, giving the possibility for unified description of inflation and dark energy [6]. There also exist more traditional mechanisms that can successfully describe the dark energy as kind of effective unusual fluid (see for example reviews [3–5]) without need to modify the gravitational theory. One of the ways that may provide a promising description of the late-time acceleration (as eventually of the inflation too) within modified gravity paradigm, is to give a mass to the spin-2 particle, the graviton, i.e. to consider massive gravity.

However, giving a mass to the graviton is not a trivial task, with the first attempts towards this goal dated back in the 40's, when Fierz and Pauli [7] attempted to describe in a linear way a massive graviton theory. Their linear theory gave rise to a discontinuity in the observable physical quantities, which discontinuity can be avoided in the context of a non-linear massive gravity theory. The non-linearities however always bring along a negative norm ghost field, which is known as the Boulware-Deser ghost, which eventually renders the theory unstable. Thus the massive gravity concept remained intangible until recently, where the interest in these massive gravity theories was renewed, since as was demonstrated in Refs. [8–10], it is possible to have a non-linear massive gravity, free of the Boulware-Deser ghost, at the decoupling limit [9] and in the full theory [10]. For some recent works on massive bigravity theories, see [11–17].

In the context of bimetric massive gravity, there are two metrics, one describing the physical Universe we observe and one background fiducial metric, and there exist solutions in which the physical metric is the Friedmann-Robertson-Walker (FRW) metric and the background metric is Minkowski metric, so non-flat, and also there exist solutions for which the background metric is dynamical and non-flat, for example a FRW one, and the physical metric is also a FRW one [17]. In principle, the massive gravity theory has many alternative descriptions, with all the descriptions

however agreeing on the fact that two metrics are necessary to exist, say $g^{\mu\nu}$ and $f^{\mu\nu}$, and the “massive” interaction term being a scalar function of $g^{\mu\nu}f_{\mu\nu}$.

Although the first attempts in massive bigravity theory made use of a flat and non-dynamical background metric $f^{\mu\nu}$, since if the fiducial metric is chosen to be a FRW one, new non-linear ghost instabilities appear [15]. However, it has been shown that [10, 11, 17] in the bimetric theories of gravity, it is possible to have a non-flat and dynamical fiducial metric, and in this case the theory is free of ghosts, again. Actually, as was demonstrated in [17], solutions of bimetric gravity in the limit where the kinetic term of the background fiducial metric vanishes, there exist massive gravity solutions compatible with a dynamical and non-flat metric. In addition, the extension of bigravity theories for the $F(R)$ gravity case has been performed in Refs. [16].

In this paper the focus is on providing a description of singular cosmological dynamics in the context of $F(R)$ bimetric gravity theory. The singularity that we shall take into account is the Type IV singularity, which is a finite-time timelike singularity [18]. According to the classification of finite-time singularities [18], the Type IV singularity is the most “harmless” one, since the Universe may smoothly pass through it without having catastrophic consequences for the observable quantities that can be defined on the spacelike three dimensional hypersurface defined at the time instance that the singularity occurs. Therefore, unlike crushing type singularities, firstly studied in a concrete way by Hawking and Penrose [19], like for example the Big-Rip singularity [20], the Type IV singularity does not affect the observables catastrophically, but affects strongly the dynamics and the slow-roll expansion of inflationary evolution [21], as was shown in [26]. Actually, the graceful exit procedure may be enhanced by the presence of a Type IV singularity, as was shown in [26]. For recent work on the Type IV singularities see [22–26], while for important earlier works on sudden singularities see [27, 28], and for alternative works on the graceful exit issue, see [29]. Since the inflationary era [21] is very important for present time observations, we shall investigate how a Type IV singularity can affect the dynamics of this era, in the context of bimetric gravity. Particularly, after presenting in brief the general bimetric gravity model we shall work on, we shall assume a quite general and simple cosmological evolution, developing a Type IV singularity, and we investigate which bimetric gravity model can successfully generate such an evolution. For the resulting model, we shall calculate the slow-roll parameters [30], and as we demonstrate, in some cases these become strongly divergent at the singularity point. We discuss the implications of these divergences in a later section where we claim that these indicate instabilities in the dynamical evolution, which in turn show that the cosmological solution which described the evolution up to the moment the singularities occur, ceases to be the final attractor of the theory, thus graceful exit from inflation may occur at the Type IV singularity. Furthermore, we study a Type IV singular version of the R^2 Starobinsky model [31], which as we show, also results in singular slow-roll parameters. Finally, we study the late-time behavior of our cosmological solutions by studying the corresponding equation of state, and in addition we propose some generalized Type IV singular cosmological evolutions, which can describe early and late-time acceleration, but also the matter domination era, by using only one model.

This paper is organized as follows: In section II we describe the theoretical framework of the $F(R)$ bimetric gravity we shall use and in section III and IV after providing the essential information for the finite-time singularities, we investigate which bimetric gravity theories can generate the singular cosmological evolutions we mentioned earlier. We also calculate the corresponding slow-roll parameters and also we investigate when these become singular at the time instance that the singularity occurs. In the end of section IV, we discuss the implications of the singularities appearing in the slow-roll parameters and consequently the implications on the inflationary dynamics. The late-time behavior and the unification of early and late-time dynamics is presented in section V, while the conclusions appear in the end of the paper.

II. BIGRAVITY ESSENTIALS

In this section we briefly review the theoretical framework of $F(R)$ bigravity and also the relevant formalism. For more details on this we refer to Ref. [16]. The general Jordan frame action of $F(R)$ bigravity contains two auxiliary fields φ and ξ , and is given below,

$$\begin{aligned}
S_F = & M_f^2 \int d^4x \sqrt{-\det f^J} \left\{ e^{-\xi} R^{J(f)} + e^{-2\xi} U(\xi) \right\} \\
& + 2m^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det g^J} \sum_{n=0}^4 \beta_n e^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{J^{-1}} f^J} \right) \\
& + M_g^2 \int d^4x \sqrt{-\det g^J} \left\{ e^{-\varphi} R^{J(g)} + e^{-2\varphi} V(\varphi) \right\} + \int d^4x \mathcal{L}_{\text{matter}}(g_{\mu\nu}^J, \Phi_i) .
\end{aligned} \tag{1}$$

In Eq. (1), $R^{(g)}$ and $R^{(f)}$ represent the scalar curvatures for the Jordan frame metrics $g_{\mu\nu}^J$ and $f_{\mu\nu}^J$, respectively. Also in the theory there exist two mass scales, the two Planck masses M_f and M_g , and we define M_{eff} to be equal to,

$$\frac{1}{M_{\text{eff}}^2} = \frac{1}{M_g^2} + \frac{1}{M_f^2}. \quad (2)$$

In addition, we also define the tensor $\sqrt{g^{-1}f}$ by using the square root of $g^{J\mu\rho}f_{\rho\nu}^J$, so that the following holds true, $\left(\sqrt{g^{-1}f}\right)^\mu{}_\rho \left(\sqrt{g^{-1}f}\right)^\rho{}_\nu = g^{\mu\rho}f_{\rho\nu}$. The symbols $e_n(X)$'s are defined for a general tensor $X^\mu{}_\nu$ in the following way,

$$\begin{aligned} e_0(X) &= 1, \quad e_1(X) = [X], \quad e_2(X) = \frac{1}{2}([X]^2 - [X^2]), \\ e_3(X) &= \frac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3]), \\ e_4(X) &= \frac{1}{24}([X]^4 - 6[X]^2[X^2] + 3[X^2]^2 + 8[X][X^3] - 6[X^4]), \\ e_k(X) &= 0 \quad \text{for } k > 4, \end{aligned} \quad (3)$$

where the trace of the tensor $X^\mu{}_\nu$: $[X] = X^\mu{}_\mu$ is denoted by $[X]$ in all the above equations and in the equations to follow, when used. By varying the action (1), with respect to φ and ξ , we can obtain algebraic equations that relate the Ricci scalars to the auxiliary fields φ and ξ . The resulting algebraic equations can, in principle, be solved algebraically with respect to the auxiliary fields φ and ξ , and upon substituting the resulting expressions into (1) we can obtain the $F(R)$ bigravity action which does not include the auxiliary scalars φ and ξ .

By conformally transforming the Jordan frame metric tensors $g_{\mu\nu}^J$ and $f_{\mu\nu}^J$ in the following way,

$$g_{\mu\nu} \rightarrow e^{-\varphi} g_{\mu\nu}^J, \quad f_{\mu\nu} \rightarrow e^\xi f_{\mu\nu}^J, \quad (4)$$

the action of Eq. (1) can be transformed as follows,

$$S_F = S_{\text{bi}} + S_\varphi + S_\xi, \quad (5)$$

$$\begin{aligned} S_{\text{bi}} &= M_g^2 \int d^4x \sqrt{-\det g} R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f} R^{(f)} \\ &\quad + 2m^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^4 \beta_n e_n \left(\sqrt{g^{-1}f} \right), \end{aligned} \quad (6)$$

$$S_\varphi = -M_g^2 \int d^4x \sqrt{-\det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right\} + \int d^4x \mathcal{L}_{\text{matter}}(e^\varphi g_{\mu\nu}, \Phi_i), \quad (7)$$

$$S_\xi = -M_f^2 \int d^4x \sqrt{-\det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + U(\xi) \right\}. \quad (8)$$

In the present paper we shall consider the simplest case, in which no matter fluids are present, so the Einstein frame action is,

$$\begin{aligned} S_{\text{bi}} &= M_g^2 \int d^4x \sqrt{-\det g} R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f} R^{(f)} \\ &\quad + 2m^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \left(3 - \text{tr} \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right). \end{aligned} \quad (9)$$

In addition we shall assume that both the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ describe a flat FRW background, and by using the conformal time t , the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, are equal to,

$$ds_g^2 = \sum_{\mu,\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu = a(t)^2 \left(-dt^2 + \sum_{i=1}^3 (dx^i)^2 \right), \quad ds_f^2 = \sum_{\mu,\nu=0}^3 f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^3 (dx^i)^2. \quad (10)$$

We also assume that the space-time is equipped with a symmetric, torsion-less, and metric compatible affine connection, the Levi-Civita connection. By defining the Hubble rates for the scale factors $a(t)$, $b(t)$ and $c(t)$, to be,

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \quad K(t) = \frac{\dot{b}(t)}{b(t)}, \quad L(t) = \frac{\dot{c}(t)}{c(t)}, \quad (11)$$

we obtain the following relations which constraint the final form of the Hubble rates,

$$cH = bK \text{ or } \frac{c\dot{a}}{a} = \dot{b}. \quad (12)$$

If $\dot{a} \neq 0$, we obtain $c = a\dot{b}/\dot{a}$ and on the other hand if $\dot{a} = 0$, we find $\dot{b} = 0$, that is, a and b must be constant and c can be arbitrarily chosen. Moreover, by redefining the scalar fields to be functions of η and ζ , that is, $\varphi = \varphi(\eta)$ and $\xi = \xi(\zeta)$ and also by identifying η and ζ with the conformal time t , that is, $\eta = \zeta = t$, we obtain the following equations,

$$\omega(t)M_g^2 = -4M_g^2(\dot{H} - H^2) - 2m^2M_{\text{eff}}^2(ab - ac), \quad (13)$$

$$\tilde{V}(t)a(t)^2M_g^2 = M_g^2(2\dot{H} + 4H^2) + m^2M_{\text{eff}}^2(6a^2 - 5ab - ac), \quad (14)$$

$$\sigma(t)M_f^2 = -4M_f^2(\dot{K} - LK) - 2m^2M_{\text{eff}}^2\left(-\frac{c}{b} + 1\right)\frac{a^3c}{b^2}, \quad (15)$$

$$\tilde{U}(t)c(t)^2M_f^2 = M_f^2(2\dot{K} + 6K^2 - 2LK) + m^2M_{\text{eff}}^2\left(\frac{a^3c}{b^2} - 2c^2 + \frac{a^3c^2}{b^3}\right), \quad (16)$$

where the functions $\omega(\eta)$, $\tilde{V}(\eta)$, $\sigma(\eta)$ and $\tilde{U}(\eta)$, have the following dependence as functions of the variable η ,

$$\omega(\eta) = 3\varphi'(\eta)^2, \quad \tilde{V}(\eta) = V(\varphi(\eta)), \quad \sigma(\zeta) = 3\xi'(\zeta)^2, \quad \tilde{U}(\zeta) = U(\xi(\zeta)). \quad (17)$$

In effect, any arbitrarily chosen cosmological evolution of the Universe, which is given by specifying the scale factors $a(t)$, $b(t)$, and $c(t)$, can be reproduced by suitably choosing the functions $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ in order these satisfy the equations (13-16), taking into account that Eq. (12) is also satisfied. In the rest of this paper we shall extensively use the formalism we presented in this section, in order to provide an $F(R)$ bigravity description of Type IV singular cosmological evolutions.

III. GENERAL SINGULAR INFLATION FROM BIGRAVITY

A. Finite-time Singularities in Cosmology

In this section we briefly review some essential information on finite-time singularities, which were first classified in Ref. [18]. The classification scheme used in Ref. [18], uses three physical quantities that can be consistently defined on any spacelike three dimensional hypersurface of constant t , namely the effective energy density, the effective pressure, the scale factor and in addition in one case it also uses the Hubble rate and its higher derivatives. All the finite-time singularities are timelike singularities, and are classified as follows,

- Type I (“Big Rip Singularity”): It is the most phenomenologically “harmful” since as the cosmic time t approaches the time instance t_s , that is, $t \rightarrow t_s$, all the physical quantities we mentioned earlier, namely, the scale factor a , the effective energy density ρ_{eff} and also the effective pressure p_{eff} become singular at the time instance $t = t_s$, that is, $a \rightarrow \infty$, $\rho_{\text{eff}} \rightarrow \infty$, and $|p_{\text{eff}}| \rightarrow \infty$ respectively. We refer the reader to Refs. [20] for more details on the Big Rip singularity.
- Type II (“Sudden Singularity”): For the Type II, as $t \rightarrow t_s$, both the effective energy density and the scale factor are finite, that is, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$, but the effective pressure becomes singular at $t = t_s$, $|p_{\text{eff}}| \rightarrow \infty$. We refer the reader to Refs. [27, 28], for more information on this singularity.
- Type III: In the Type III case, as $t \rightarrow t_s$, both the effective pressure and the effective energy density diverge, $\rho_{\text{eff}} \rightarrow \infty$ and $|p_{\text{eff}}| \rightarrow \infty$, but the scale factor does not diverge, that is, $a \rightarrow a_s$.
- Type IV: This singularity is the one which concern us in this paper, since it is the most “mild” regarding the phenomenological implications. In this case all the aforementioned physical quantities are finite at $t = t_s$, that is, $a \rightarrow a_s$, $\rho_{\text{eff}} \rightarrow \rho_s$, $|p_{\text{eff}}| \rightarrow p_s$, but the higher derivatives of order $n \geq 2$ of the Hubble rate diverge, but the Hubble rate is finite of course. We refer the reader to [22–26] for some recent studies on the cosmological implications of the Type IV singularity.

As we already mentioned, the focus in this paper will be on the Type IV singularity, which we study in the Jordan frame.

B. Singular Inflation: The Bigravity Description

Now we proceed to find how singular inflation can be realized in bigravity. First note that, as we already discussed before Eq. (5), the physical metric, which is determined in the frame where the scalar does not directly couple with matter, is given by multiplying the scalar field and the metric in the Einstein frame in Eq. (10),

$$g_{\mu\nu}^J = e^{\varphi} g_{\mu\nu}, \quad (18)$$

and hereafter we shall call $g_{\mu\nu}^J$ the Jordan frame metric. By using the cosmological time \tilde{t} in the Jordan frame, the FRW metric is assumed to be equal to,

$$ds^2 = -d\tilde{t}^2 + \tilde{a}(\tau)^2 \sum_{i=1}^3 (dx^i)^2. \quad (19)$$

In order to have a Type IV singular evolution, we assume that the Hubble rate $\tilde{H}(\tilde{t}) \equiv \frac{1}{\tilde{a}(\tilde{t})} \frac{d\tilde{a}(\tilde{t})}{d\tilde{t}}$, behaves as follows,

$$\tilde{H}(\tilde{t}) \sim H_0 + H_1 |\tilde{t} - \tilde{t}_s|^\gamma. \quad (20)$$

Then, according to classification of finite-time singularities we presented earlier, when $\gamma \leq -1$, the cosmological evolution develops a Type I singularity, which is nothing but the Big Rip singularity [20], while when $-1 < \gamma < 0$, this case corresponds to a Type III singularity. When $0 < \gamma < 1$, the cosmological evolution develops a Type II singularity, and finally, in the case that $1 < \gamma$ and γ is assumed to be a non-integer number, this leads to a Type IV finite-time singularity. It is the last case that we are interested in, so hereafter we assume that $\gamma > 1$. Notice that when $\gamma \neq -1$, the scale factor $\tilde{a}(\tilde{t})$ corresponding to (20) is given by,

$$a(\tilde{t}) \propto e^{H_0 \tilde{t} + \text{sign}(\tilde{t} - \tilde{t}_s) \frac{H_1}{\gamma+1} |\tilde{t} - \tilde{t}_s|}. \quad (21)$$

In the above equation, the symbol sign is defined as follows,

$$\text{sign}(x) = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \end{cases}. \quad (22)$$

Consequently, Eq. (21) indicates that when $\gamma > -1$, the scale factor is finite even at the point of the singularity $\tilde{t} = \tilde{t}_s$, and this also covers the Type IV case. Therefore, even if we consider the conformal time t , the FRW metric is given by,

$$ds^2 = a_J(t)^2 \left(-dt^2 + \tilde{a}(\tau)^2 \sum_{i=1}^3 (dx^i)^2 \right), \quad (23)$$

and in effect, we find that $t \sim \tilde{t}$ owing to the fact that $d\tilde{t} = \tilde{a}(\tilde{t})dt$ and also since $\tilde{a}(\tilde{t}) \sim 1$ at the singularity $\tilde{t} = \tilde{t}_s$. By looking at Eq. (23), we conclude that,

$$a_J(t) = \tilde{a}(\tilde{t}). \quad (24)$$

and this indicates that near the singularity, the Hubble rate in terms of $a_J(t)$ and of the conformal time t behaves as in (21), that is,

$$H_J(t) \equiv \frac{1}{a_J(t)} \frac{da_J(t)}{dt} \sim H_0 + H_1 |t - t_s|^\gamma. \quad (25)$$

We should also note that,

$$H_J(t) = H(t) + \frac{\dot{\varphi}(t)}{2}. \quad (26)$$

with the $H(t)$ appearing in Eq. (26), being defined after Eq. (10). Owing to the fact that the space-time is described by the metric $g_{\mu\nu}^J$, the functions $a(t)$, $b(t)$ and $c(t)$ are not always directly related with the expansion of the Universe.

Consequently, we may choose the scale factors $a(t)$, $b(t)$ and $c(t)$ in a way consistent with Eq. (12). In the following, we shall assume that $a(t) = b(t) = 1$, so that, Eq. (26) indicates that,

$$H_J(t) = \frac{\dot{\varphi}(t)}{2}. \quad (27)$$

Then Eqs. (13), (14), (15), and (16) are simplified as follows,

$$\omega(t)^2 M_g^2 = 12M_g^2 H_J^2 = m^2 M_{\text{eff}}^2 (c - 1), \quad (28)$$

$$\tilde{V}(t) M_g^2 = m^2 M_{\text{eff}}^2 (1 - c) = -6M_g^2 H_J^2, \quad (29)$$

$$\sigma(t) M_f^2 = 2m^2 M_{\text{eff}}^2 (c - 1) = 12M_g^2 H_J^2, \quad (30)$$

$$\tilde{U}(t) M_f^2 = m^2 M_{\text{eff}}^2 c (1 - c) = -6M_g^2 H_J^2 \left(1 + \frac{6H_J^2}{m^2 M_{\text{eff}}^2} \right). \quad (31)$$

The Eq. (28) can be solved explicitly with respect to c , and it yields,

$$c(t) = 1 + \frac{6H_J^2}{m^2 M_{\text{eff}}^2}. \quad (32)$$

Then, by using (25), we find that the model generating Type II, III, or IV singularities is given by

$$\omega(t)^2 M_g^2 = 12M_g^2 (H_0 + H_1 |t - t_s|^\gamma)^2, \quad (33)$$

$$\tilde{V}(t) M_g^2 = -6M_g^2 (H_0 + H_1 |t - t_s|^\gamma)^2, \quad (34)$$

$$\sigma(t) M_f^2 = 12M_g^2 (H_0 + H_1 |t - t_s|^\gamma)^2, \quad (35)$$

$$\tilde{U}(t) M_f^2 = -6M_g^2 (H_0 + H_1 |t - t_s|^\gamma)^2 \left(1 + \frac{6(H_0 + H_1 |t - t_s|^\gamma)^2}{m^2 M_{\text{eff}}^2} \right). \quad (36)$$

and therefore we find that effectively, $c(t)$ is equal to,

$$c(t) = 1 + \frac{6(H_0 + H_1 |t - t_s|^\gamma)^2}{m^2 M_{\text{eff}}^2}. \quad (37)$$

Let us now proceed to the calculation of the slow-roll parameters for the Type IV singular evolution model, and we shall calculate in detail the slow-roll parameters ϵ , η and ξ . When we use the cosmological time t in (19) and the e-foldings N defined by $a = a_0 e^N$ with a constant a_0 , we can express the slow-roll parameters by using \tilde{H} as follows,

$$\begin{aligned} \epsilon &= -\frac{\tilde{H}(N)}{4\tilde{H}'(N)} \left[\frac{6\frac{\tilde{H}'(N)}{\tilde{H}(N)} + \frac{\tilde{H}''(N)}{\tilde{H}(N)} + \left(\frac{\tilde{H}'(N)}{\tilde{H}(N)} \right)^2}{3 + \frac{\tilde{H}'(N)}{\tilde{H}(N)}} \right]^2, \\ \eta &= -\frac{1}{2} \left(3 + \frac{\tilde{H}'(N)}{\tilde{H}(N)} \right)^{-1} \left[9\frac{\tilde{H}'(N)}{\tilde{H}(N)} + 3\frac{\tilde{H}''(N)}{\tilde{H}(N)} + \frac{1}{2} \left(\frac{\tilde{H}'(N)}{\tilde{H}(N)} \right)^2 - \frac{1}{2} \left(\frac{\tilde{H}''(N)}{\tilde{H}'(N)} \right)^2 + 3\frac{\tilde{H}'''(N)}{\tilde{H}'(N)} + \frac{\tilde{H}''''(N)}{\tilde{H}'(N)} \right], \\ \xi^2 &= \frac{6\frac{\tilde{H}'(N)}{\tilde{H}(N)} + \frac{\tilde{H}''(N)}{\tilde{H}(N)} + \left(\frac{\tilde{H}'(N)}{\tilde{H}(N)} \right)^2}{4 \left(3 + \frac{\tilde{H}'(N)}{\tilde{H}(N)} \right)^2} \left[3\frac{\tilde{H}(N)\tilde{H}'''(N)}{\tilde{H}'(N)^2} + 9\frac{\tilde{H}'(N)}{\tilde{H}(N)} - 2\frac{\tilde{H}(N)\tilde{H}''(N)\tilde{H}'''(N)}{\tilde{H}'(N)^3} + 4\frac{\tilde{H}''(N)}{\tilde{H}(N)} \right. \\ &\quad \left. + \frac{\tilde{H}(N)\tilde{H}''(N)^3}{\tilde{H}'(N)^4} + 5\frac{\tilde{H}'''(N)}{\tilde{H}'(N)} - 3\frac{\tilde{H}(N)\tilde{H}''(N)^2}{\tilde{H}'(N)^3} - \left(\frac{\tilde{H}''(N)}{\tilde{H}'(N)} \right)^2 + 15\frac{\tilde{H}''(N)}{\tilde{H}'(N)} + \frac{\tilde{H}(N)\tilde{H}''''(N)}{\tilde{H}'(N)^2} \right]. \end{aligned} \quad (38)$$

In addition, the relations between $\tilde{H}(N)$ and $H_J(t)$ are given below,

$$\tilde{H}(N) = \frac{H_J(t)}{a(t)}, \quad \tilde{H}'(N) = \frac{1}{a(t)H_J(t)} \left(-H_J(t)^2 + \dot{H}_J(t) \right),$$

$$\begin{aligned}
\tilde{H}(N)'' &= \frac{1}{a(t)H_J(t)} \left(H_J(t)^2 - 2\dot{H}_J(t) - \frac{(\dot{H}_J(t))^2}{H_J(t)^2} + \frac{\ddot{H}_J(t)}{H_J(t)^2} \right), \\
\tilde{H}(N)''' &= \frac{1}{a(t)H_J(t)} \left(-H_J(t)^2 + 5\dot{H}_J(t) - \frac{(\dot{H}_J(t))^2}{H_J(t)^2} - \frac{3\ddot{H}_J(t)}{H_J(t)^2} - \frac{2\dot{H}_J(t)\ddot{H}_J(t)}{H_J(t)^3} + \frac{\ddot{\ddot{H}}_J(t)}{H_J(t)^2} \right), \\
\tilde{H}(N)'''' &= \frac{1}{a(t)H_J(t)} \left(H_J(t)^2 - 8\dot{H}_J(t) + \frac{6(\dot{H}_J(t))^2}{H_J(t)^2} + \frac{(\dot{H}_J(t))^3}{H_J(t)^3} + \frac{8\ddot{H}_J(t)}{H_J(t)^2} - \frac{2(\ddot{H}_J(t))^2}{H_J(t)^4} \right. \\
&\quad \left. + \frac{2(\dot{H}_J(t))^2\ddot{H}_J(t)}{H_J(t)^5} - \frac{4\ddot{\ddot{H}}_J(t)}{H_J(t)^2} - \frac{3\dot{H}_J(t)\ddot{\ddot{H}}_J(t)}{H_J(t)^4} + \frac{\ddot{\ddot{\ddot{H}}}_J(t)}{H_J(t)^2} \right). \tag{39}
\end{aligned}$$

Having these at hand, we may evaluate the slow-roll parameters near the Type IV singularity at $t \sim t_s$. Recall that since the singularity is a Type IV one, the parameter γ has to obey $\gamma > 1$ and also γ must not an integer in (25). Therefore, by using the above equations, when $1 < \gamma < 2$, we find,

$$\begin{aligned}
\tilde{H} &\sim \frac{H_0}{a(t_s)}, \quad \tilde{H}' \sim -\frac{H_0}{a(t_s)}, \quad \tilde{H}'' \sim \frac{H_1\gamma(\gamma-1)}{a(t_s)H_0^3} |t-t_s|^{\gamma-2}, \\
\tilde{H}''' &\sim \frac{H_1\gamma(\gamma-1)(\gamma-2)}{\text{sign}} (t-t_s) a(t_s) H_0^3 |t-t_s|^{\gamma-3}, \quad \tilde{H}'''' \sim \frac{H_1\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{a(t_s)H_0^3} |t-t_s|^{\gamma-4}. \tag{40}
\end{aligned}$$

In the case that $2 < \gamma < 3$, the term \tilde{H}'' in (19) is replaced by the following expression,

$$\tilde{H}'' \rightarrow \frac{H_0}{a(t_s)}, \tag{41}$$

while in the case that $3 < \gamma < 4$, in addition to the term \tilde{H}'' , also \tilde{H}''' should be taken into account, and it is replaced by,

$$\tilde{H}''' \rightarrow -\frac{H_0}{a(t_s)}. \tag{42}$$

Finally, when $\gamma > 4$ and γ is not an integer, the term \tilde{H}'''' is replaced by the following expression,

$$\tilde{H}'''' \rightarrow \frac{H_0}{a(t_s)}. \tag{43}$$

Consequently, when $1 < \gamma < 2$, we find that the slow-roll parameters become,

$$\begin{aligned}
\epsilon &\sim -\frac{H_1\gamma(\gamma-1)}{16H_0^4} |t-t_s|^{\gamma-2}, \quad \eta \sim \frac{H_1\gamma(\gamma-1)(\gamma-2)}{4H_0^4} \text{sign}(t-t_s) |t-t_s|^{\gamma-3}, \\
\xi^2 &\sim -\frac{5H_1\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{16H_0^4} |t-t_s|^{\gamma-4}, \tag{44}
\end{aligned}$$

which are singular at $t = t_s$. In the case $2 < \gamma < 3$, ϵ becomes finite even at $t = t_s$ and we find that the slow-roll parameter is equal to,

$$\epsilon = -\frac{1}{4}. \tag{45}$$

In the case $3 < \gamma < 4$, the slow-roll parameter η also becomes finite and it is equal to,

$$\eta = 2. \tag{46}$$

In addition, when $\gamma > 4$ and γ is not an integer, all of ϵ , η , and ξ^2 are finite and we find that,

$$\xi^2 = 4. \tag{47}$$

Hence, we demonstrated that the slow-roll indices may be singular in the case that a Type IV singularity occurs in the cosmological evolution. But what does exactly this singularity indicates? As was shown in Refs. [26], singularities in the slow-roll indices indicate dynamical instability of the inflationary process. This does not by no mean that the spectral observational indices of inflation become infinite. As was demonstrated in [30], the slow-roll indices ϵ and η are first order terms in the slow-roll expansion. So when these become large (of order ~ 1), the slow-roll expansion breaks down and inflation ends. This is exactly what happens in our case, and since the slow-roll parameters diverge at $t = t_s$, this means that inflation should end there in an abrupt way. We shall further discuss this issue in a later section, in more detail, but for illustrative reasons, let us assume that we calculate the inflationary observational indices, and we assume that we use the singular expressions for the slow-roll parameters we found in this section. For simplicity, let us assume that the definition of the inflationary indices is that of a canonical scalar coupled with gravity, so the spectral index of the primordial curvature fluctuations n_s , the tensor-to-scalar ratio r , and the associated running of the spectral index α_s are given by

$$n_s \sim 1 - 6\epsilon + 2\eta, \quad r \sim 16\epsilon, \quad \alpha_s \sim 16\epsilon\eta - 24\epsilon^2 - 2\xi^2. \quad (48)$$

Although there is no reason that the expressions in (48) can be justified for the $F(R)$ bigravity model in this paper, we may evaluate the quantities n_s , r , and α_s by using (48), by assuming the end of the inflation is given by $t = t_f$ but also that $t_f \sim t_s$. Then, in the case that $1 < \gamma < 2$, we find,

$$\begin{aligned} n_s &\sim \frac{H_1\gamma(\gamma-1)(\gamma-2)}{2H_0^4} \text{sign}(t_f - t_s) |t_f - t_s|^{\gamma-3}, \quad r \sim -\frac{H_1\gamma(\gamma-1)}{H_0^4} |t_f - t_s|^{\gamma-2}, \\ \alpha_s &\sim \frac{5H_1\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{8H_0^4} |t_f - t_s|^{\gamma-4}, \end{aligned} \quad (49)$$

and correspondingly in the case that $2 < \gamma < 3$, we obtain,

$$\begin{aligned} n_s &\sim \frac{H_1\gamma(\gamma-1)(\gamma-2)}{2H_0^4} \text{sign}(t_f - t_s) |t_f - t_s|^{\gamma-3}, \quad r \sim -4, \\ \alpha_s &\sim \frac{5H_1\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{8H_0^4} |t_f - t_s|^{\gamma-4}. \end{aligned} \quad (50)$$

Furthermore, in the case of $3 < \gamma < 4$, we obtain,

$$n_s \sim \frac{13}{2}, \quad r \sim -4, \quad \alpha_s \sim \frac{5H_1\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{8H_0^4} |t_f - t_s|^{\gamma-4}. \quad (51)$$

and finally, in the case that $\gamma > 4$, we get,

$$n_s \sim \frac{13}{2}, \quad r \sim -4, \quad \alpha_s \sim -\frac{35}{2}. \quad (52)$$

By looking at Eqs. (49), (50), and (51), we may loosely say that the spectral inflationary indices may diverge at the singularity, but this is not true. This is owing to the fact that the spectral indices are given in terms of the slow-roll parameters only in the case the slow-roll approximation holds true, so when $\epsilon, \eta \ll 1$, which is not the case when a singularity appears in the slow-roll parameters. It can be true that $\epsilon, \eta \ll 1$ long before the singularity, but at the singularity, the inflationary indices cannot be written in terms of the slow-roll parameters since the perturbation slow-roll expansion breaks down at the singularity.

Notice, however, that in the case of $1 < \gamma < 2$, by suitably choosing the parameters H_0 , H_1 , γ , and $t_f - t_s$, the resulting expression can be compatible with the 2015 Planck report [32] which restricts the inflationary indices as follows,

$$n_s = 0.9644 \pm 0.0049, \quad r < 0.10, \quad \alpha_s = -0.0057 \pm 0.0071. \quad (53)$$

IV. SINGULAR NEARLY R^2 INFLATION FROM BIGRAVITY

In the previous section we discussed how a Type IV singular cosmological evolution can be produced by bigravity, and as we demonstrated this leads to slow-roll parameters that may contain singularities. These singularities can be valuable from a physical point of view, since these indicate strong instabilities of the dynamical evolution of the physical system. In this section we shall study a Type IV singular version of the Starobinsky R^2 inflation model

[31], to which we refer as “singular R^2 inflation” model, hereafter. Particularly, we shall incorporate the Type IV singularity in the Hubble rate that corresponds to the Jordan frame R^2 inflation model, the action of which is,

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R + \frac{1}{6M^2} R^2 \right), \quad (54)$$

with the parameter satisfying $M \gg 1$. Note that our approach is not directly related to the Jordan frame $F(R)$ theory, since we shall try to generate the Hubble rate that the vacuum $F(R)$ theory of Eq. (54) generates by using the bigravity theoretical framework. To this end, let us calculate the Hubble rate that is generated by the $F(R)$ gravity of Eq. (54), so that we use it as a basis for our calculations. The FRW equation corresponding to the $F(R)$ gravity of Eq. (54) is equal to,

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{M^2}{2} H = -3H\dot{H}, \quad (55)$$

and owing to the fact that during inflation, the terms \ddot{H} and \dot{H} can be considered subdominant, the resulting Hubble rate can be found by solving the differential equation of Eq. (55), and the result is approximately equal to,

$$H(t) \simeq H_i - \frac{M^2}{6} (t - t_i). \quad (56)$$

In the above equation, t_i is the initial time instance that we assume inflation starts and the parameter H_i is the corresponding value of the Hubble rate at the time instance $t = t_i$.

The purpose of this section is twofold, firstly we shall see how a singular R^2 inflation can be generated by a bigravity theory and secondly we shall see what are the new qualitative features of this singular R^2 inflation model, focusing on the inflationary indices. As we shall demonstrate, the “harmless” Type IV singularity may generate strong dynamical instabilities manifested in the slow-roll parameters. The simplest way to add a Type IV singularity in the R^2 inflation Hubble rate of Eq. (56), is the following,

$$H(t) \simeq H_i - \frac{M^2}{6} (t - t_i) + f_0 (t - t_s)^\gamma, \quad (57)$$

where we assumed that the Type IV singularity occurs at $t = t_s$, and also that $\gamma > 1$, in order for the cosmological evolution to develop a Type IV singularity. In addition, in order the effects of the singularity on the cosmological evolution are small, we further assume that $H_i \gg f_0$, $M \gg f_0$ and also that $f_0 \ll 1$. In effect, the singularity term $\sim f_0 (t - t_s)^\gamma$, is significantly smaller compared to the other two terms of Eq. (57). Consequently, the effect of the singularity on the Hubble rate is practically insignificant, when $t \rightarrow t_s$.

Before going into the detailed calculation of the slow-roll indices, let us see which bigravity model can produce the cosmological evolution of Eq. (57). We can easily see that if we determine the functions $\omega(t)$, $V(t)$, $\sigma(t)$ and $U(t)$ which appear in Eq. (33), which by substituting the Hubble rate of Eq. (57) in Eq. (33), we easily get,

$$\begin{aligned} \omega(t) &= 2\sqrt{3} \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right), \\ \tilde{V}(t) &= -6 \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right)^2, \\ \sigma(t) &= \frac{12M_g^2 \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right)^2}{M_f^2}, \\ \tilde{U}(t) &= - \frac{6M_g^2 \left(1 + \frac{6 \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right)^2}{m^2 M_{\text{eff}}^2} \right) \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right)^2}{M_f^2}. \end{aligned} \quad (58)$$

and note that we took into account Eq. (24), so the above relations hold true near the singularity, where the conformal time is nearly identical with the cosmic time. We proceed to the calculation of the slow-roll parameters, and by using Eqs. (38) and (57), we can obtain the exact form of the slow-roll indices ϵ and η , which appear in the Appendix. The parameter ξ is very large to be presented in detail, so we presented in the Appendix only the terms that can be singular at the Type IV singularity time instance $t = t_s$. As it can be seen in the Appendix, the slow-roll parameters

can be singular at $t = t_s$ for various values of the parameter γ , when $\gamma > 1$. To see this explicitly, we present the approximate form of the slow-roll parameters for $t \rightarrow t_s$, starting with the parameter ϵ , which reads,

$$\epsilon \simeq - \frac{3f_0^2(t-t_s)^{-4+2\gamma}(-1+\gamma)^2\gamma^2}{2(-6H_i + M^2(-1+t-t_i))(H_i + \frac{1}{6}M^2(-t+t_i))^5 \left(3 - \frac{36(6H_i+M^2(1-t+t_i))}{(6H_i+M^2(-t+t_i))^3}\right)^2}. \quad (59)$$

As it can be seen, when $1 < \gamma < 2$, the term $\sim (t-t_s)^{2(-2+\gamma)}$, becomes singular at $t = t_s$ and therefore the slow-roll parameter ϵ becomes singular too. This infinite instability clearly indicates that the dynamics of inflation are strongly disturbed at the infinite instability time instance. We shall thoroughly discuss this issue in the next section, so we refer from going into further details on this issue for the moment. In the case, $\gamma > 2$, the slow-roll parameter ϵ becomes,

$$\begin{aligned} \epsilon \simeq & - \frac{(6H_i + M^2(-t+t_i))^2 (72M^2 + (36 + 36H_i^2 - 12H_i(18 + M^2(t-t_i)))}{1119744(-6H_i + M^2(-1+t-t_i))(H_i + \frac{1}{6}M^2(-t+t_i))^5 \left(3 - \frac{36(6H_i+M^2(1-t+t_i))}{(6H_i+M^2(-t+t_i))^3}\right)^2} \\ & - \frac{12M^2(-2+3t-3t_i) + M^4(t-t_i)^2 (6H_i + M^2(-t+t_i))^2}{1119744(-6H_i + M^2(-1+t-t_i))(H_i + \frac{1}{6}M^2(-t+t_i))^5 \left(3 - \frac{36(6H_i+M^2(1-t+t_i))}{(6H_i+M^2(-t+t_i))^3}\right)^2}, \end{aligned} \quad (60)$$

so the Type IV singularity has no effect on the slow-roll parameter ϵ in this case. Accordingly, the slow-roll index η for the Hubble rate (57) is given in detail in the Appendix, and below we quote the approximate form of η near the singularity at $t \simeq t_s$, by keeping only the terms that contain singularities,

$$\begin{aligned} \eta_{sing} \simeq & \frac{1}{4 \left(3 - \frac{36(6H_i+M^2(1-t+t_i))}{(6H_i+M^2(-t+t_i))^3}\right)} \left(- \frac{3f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(H_i + \frac{1}{6}M^2(-t+t_i))^4} \right. \\ & - \frac{36f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(-6H_i + M^2(-1+t-t_i))(H_i + \frac{1}{6}M^2(-t+t_i))^2} + \frac{f_0^2(t-t_s)^{-4+2\gamma}(-1+\gamma)^2\gamma^2}{(H_i + \frac{1}{6}M^2(-t+t_i))^8} \\ & + 12 \left(\frac{72f_0M^2(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^3 - 6H_i + M^2(-1+t-t_i)} \right. \\ & \left. \left. - \frac{36f_0(t-t_s)^{-3+\gamma}(2+3t-3t_s-\gamma)(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^2 - 6H_i + M^2(-1+t-t_i)} \right) \right). \end{aligned} \quad (61)$$

In this case, the terms that contain singularities are the ones listed below,

- When $1 < \gamma < 2$, the singular terms are, $\sim (t-t_s)^{-2+\gamma}$, $\sim (t-t_s)^{-3+\gamma}$, $\sim (t-t_s)^{-4+2\gamma}$
- When $2 < \gamma < 3$, the singular terms are, $\sim (t-t_s)^{-3+\gamma}$

When $\gamma > 3$ no singular terms occur, so the slow-roll parameter η reads,

$$\begin{aligned} \eta \simeq & - \frac{1}{4 \left(3 - \frac{36(6H_i+M^2(1-t+t_i))}{(6H_i+M^2(-t+t_i))^3}\right)} \left(- \frac{648(6H_i + M^2(1-t+t_i))}{(6H_i + M^2(-t+t_i))^3} + \frac{36(6H_i + M^2(1-t+t_i))^2}{(6H_i + M^2(-t+t_i))^4} \right. \\ & + \frac{12 \left(-\frac{5M^2}{6} - (H_i + \frac{1}{6}M^2(-t+t_i))^2 - \frac{M^4}{(6H_i+M^2(-t+t_i))^2} \right)}{-6H_i + M^2(-1+t-t_i)} \\ & - \frac{\left(\frac{M^2}{3} + (H_i + \frac{1}{6}M^2(-t+t_i))^2 - \frac{M^4}{(6H_i+M^2(-t+t_i))^2} \right)^2}{(H_i + \frac{1}{6}M^2(-t+t_i))^4} \\ & + \frac{3 \left(\frac{M^2}{3} + (H_i + \frac{1}{6}M^2(-t+t_i))^2 - \frac{M^4}{(6H_i+M^2(-t+t_i))^2} \right)}{(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)\gamma)^2} \\ & \left. + \frac{36 \left(\frac{M^2}{3} + (H_i + \frac{1}{6}M^2(-t+t_i))^2 - \frac{M^4}{(6H_i+M^2(-t+t_i))^2} \right)}{-6H_i + M^2(-1+t-t_i) - 6f_0(t-t_s)^{-1+\gamma}(t-t_s-\gamma)} \right). \end{aligned} \quad (62)$$

As in the case of the slow-roll parameter, the singularities on the slow-roll index clearly indicate that the dynamics of inflation as these are quantified in terms of the slow-roll index η , are in some way interrupted, or more correctly become unstable at the singularity point. Let us recall what the slow-roll parameters ϵ and η indicate for the inflationary dynamics. First, these parameters are of first order in the slow-roll perturbative expansion, with the slow-roll parameter ϵ indicating if inflation occurs, and the second slow-roll parameter measuring how much does inflation lasts [26, 30]. Therefore, the infinite instability of these parameters at some time instance clearly indicates that the dynamics is interrupted. However, the dynamics might also be interrupted at higher orders in the Hubble slow-roll expansion, therefore this could be an indication that inflation ends at the singularity. This issue is very important and needs to be further discussed, so we defer this discussion to the next section. Before closing this section, let us note that we calculated the slow-roll parameter ξ , for the Hubble evolution of Eq. (57), but the resulting expression is too large to quote it here, and even the approximate expression near the singularity is quite lengthy. So in the Appendix we have included all the terms that contain singularities in the case of the slow-roll parameter ξ . Below we quote the singular terms that the parameter ξ contains, for various values of the parameter γ .

- When $1 < \gamma < 2$, the singular terms are, $\sim (t - t_s)^{-6+3\gamma}$, $(t - t_s)^{-4+\gamma}$, $\sim (t - t_s)^{-2+\gamma}$, $\sim (t - t_s)^{-3+\gamma}$, $\sim (t - t_s)^{-4+2\gamma}$
- When $2 < \gamma < 3$, the singular terms are, $\sim (t - t_s)^{-3+\gamma}$, $(t - t_s)^{-4+\gamma}$
- When $3 < \gamma < 4$, the only singular term is $(t - t_s)^{-4+\gamma}$.

It is conceivable that the parameter ξ is free of singularities when $\gamma > 4$. In Table I, we gathered all the results, regarding the singularities of the slow-roll parameters at $t = t_s$, for various values of the parameter γ . As it can be seen, when $\gamma > 4$, the slow-roll parameters are free of singularities. As a concluding remark, we need to note that although that the singularity remains unnoticed when one considers the Hubble rate and all the physical quantities that can be defined on the three-dimensional spacelike hypersurface $t = t_s$, it has a strong effect on the dynamics of the cosmological evolution, drastically affecting the slow-roll parameters, which control the dynamics of inflation.

TABLE I: Singularity Structure of the slow-roll parameters ϵ , η and ξ for various values of the parameter γ

Slow-roll Parameter	$1 < \gamma < 2$	$2 < \gamma < 3$	$3 < \gamma < 4$
ϵ	Singular	Non-singular	Non-singular
η	Singular	Singular	Non-singular
ξ	Singular	Singular	Singular

A. Graceful Exit via Dynamical Instabilities-A Critical Discussion

In the previous two sections we demonstrated how a Type IV singularity can be incorporated in the cosmological evolution of the Universe, and we investigated which bigravity theory can successfully generate such an evolution. Particularly we studied two cosmological models, one corresponding to the Hubble rate of Eq. (20) and another one with the Hubble rate being the one of Eq. (57). The latter model is a singular deformation of the R^2 inflation model [31]. In both cases, the Type IV singularity in the Hubble rate is generated by a term $\sim (t - t_s)^\gamma$, where the parameter γ is assumed to be $\gamma > 1$, so that a Type IV singularity occurs. As we showed, the presence of the singularity plays no role in the cosmological evolution, since the Hubble rate and all the physical quantities that can be defined on the three dimensional spacelike hypersurface $t = t_s$ are finite. However, the effects of the singularity are quite severe when the slow-roll parameters are taken into account. Indeed, by calculating these, we demonstrated that these become infinite for certain values of the parameter γ . Recall that the slow-roll parameters determine the dynamics of inflation [30], and actually these indicate if inflation occurs in the first place and how long it lasts. Particularly, the slow-roll parameter ϵ determines if inflation begins, and in order slow-roll evolution occurs, it must obey $\epsilon \ll 1$, while the parameter η , determines how long inflation lasts. The presence of singularities in these slow-roll parameter clearly indicates that the dynamics of inflation becomes unstable at the singularity point. This infinite instability shows that the dynamical evolution is abruptly interrupted at the singularity and therefore the inflationary attractor that described the dynamical system up to that point ceases to be the final attractor of the theory. Consequently, the presence of singularities indicates that at the point of singularities, the graceful exit from inflation occurs. Of course it is conceivable that the singularities solely do not generate graceful exit from inflation, but they provide clear information that the graceful exit occurs at that point. The actual mechanism for graceful exit should be some curvature perturbation instability [26, 33] or a tachyonic instability [34], but nevertheless, this graceful exit

should occur at the singularity point $t = t_s$. For some relevant studies of the infinite instability that the Type IV singularity generates, see Ref. [26]. We need to stress that the slow-roll parameters ϵ and η , are of first order in the slow-roll perturbative expansion [30], and therefore the presence of instabilities at these parameters generates the question whether inflation actually ended at a higher order in the perturbative slow-roll expansion. Actually, if higher order slow-roll parameters also become infinite at the Type IV singularity, this shows that the perturbative slow-roll expansion breaks at higher order and hence, this indicates that inflation ends since the perturbative expansion breaks at a higher order. In principle we could go towards this direction and calculate these higher order slow-roll parameters, but their complicated form would make the presentation of the paper unnecessary complicated, so we defer from going into detail. But it can be easily checked that the qualitative picture we just discussed, indeed holds true.

Before closing this section, we need to stress another interesting possibility, related to the singular R^2 Starobinsky inflation model. In the ordinary Starobinsky model, graceful exit from inflation occurs when the slow-roll parameter ϵ becomes of the order ~ 1 , and suppose that this happens when $t = t_f$. As it can be checked, the second slow-roll parameter η is finite at the moment that ϵ becomes of the order ~ 1 [26], that is, at the time instance $t = t_f$. In the singular R^2 model, regardless when ϵ becomes of the order $\epsilon \sim 1$, at the singularity, the slow-roll parameters become infinite and therefore inflation might end at the singularity in a more abrupt way, since the inflationary dynamics are severely interrupted at the singular point. Hence, we may have two interesting possibilities, either inflation ends at $t = t_s$, with $t_s < t_f$, and therefore earlier from the time instance $t = t_f$, or inflation ends at $t = t_f = t_s$, so in this case inflation ends at the time that the ordinary R^2 inflation exits from inflation, with the difference being that, in the singular Starobinsky case, inflation ends more abruptly. For a thorough discussion on these issues, we refer the reader to Refs. [26]. Finally, let us note that in principle someone would claim that the observational indices n_s and r become infinite at the singularities, this however is not true. This is owing to the fact that the spectral index of primordial curvature perturbations can be written in terms of the slow-roll parameters ϵ and η only in the case that these satisfy the slow-roll condition $\epsilon, \eta \ll 1$. In addition, these are calculated at the time that the quantum fluctuations of the comoving scalar curvature exit the horizon, that is, when the corresponding wavelengths become of the order of the Hubble radius $r_h = \frac{1}{a(t)H(t)}$, which occurs much more earlier than the graceful exit from inflation. Therefore, no infinity can occur at the observational indices and the interpretation of the singularities appearance in the slow-roll indices clearly affects only the graceful exit from inflation era. So the corresponding wavelengths that exit the horizon at the moment that graceful exit occurs, are irrelevant to present time observations. This is because the only modes that are relevant at present time are the ones with wavelength equal to the Hubble radius at the moment of horizon crossing long before the graceful exit, which re-enter the horizon after the reheating of the Universe. Hence, no infinities at the observational indices occur, and these occur only at the slow-roll parameters. As we evinced, this behavior shows that the dynamical evolution becomes unstable, but all the physical quantities are finite.

V. LATE-TIME BEHAVIOR OF THE SINGULAR INFLATION MODELS

The singular inflation models we presented in the previous sections can potentially have a quite interesting late-time behavior, from a phenomenological point of view. In this section we properly modify the models we worked out in the previous section, so that to achieve the unification of early-time and late-time acceleration with the same model. Note that assuming the perfect fluid form [2], the effective equation of state (EoS), for a general bigravity model is given by,

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}, \quad (63)$$

As a first example that late-time and early-time acceleration occurs, we can think as follows: In the metric of Eq. (23), if the scale factor $a_J(t)$ behaves as $a_J(t)^2 = \frac{l^2}{t^2}$ or equivalently if the Hubble rate behaves as $H_J(t) = -\frac{1}{t}$, then the metric describes a de Sitter evolution of the Universe. Note that the parameter l appearing in the scale factor $a_J(t)$ and in the Hubble rate $H_J(t)$, is a constant of length dimension. In the case that the scale factor behaves as $\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$ or equivalently the Hubble rate behaves as $H_J(t) = -\frac{n}{t}$ with $n \neq 1$, if $0 < n < 1$, then the metric corresponds to a phantom evolution of the Universe. In addition, in the case that $n > 1$, the Universe's evolution is a quintessential acceleration, and in the case that $n < 0$, the Universe decelerating. We should note that in order for the Universe to be expanding, we should have $t < 0$ when $n > 0$, and in addition, $t > 0$ when $n < 0$. Then, if we consider Einstein gravity coupled with a perfect fluid with its EoS parameter being equal to,

$$w = -\frac{1}{3} \left(2 + \frac{1}{n} \right), \quad (64)$$

then we find that $n = -\frac{1}{3w+2}$. Hence, in the case that the perfect fluid describes collisionless dust, we must have $w = 0$ and therefore, n must be $n = -\frac{1}{2}$.

We now turn our focus on another example with interesting early-time and late-time acceleration, for which the unification of these two accelerating eras can be achieved. We will assume that a Type IV is incorporated in the model, so the Hubble rate is equal to,

$$H(t) = (H_0 + H_1 |t - t_s|^\gamma) \frac{e^{-\frac{t-t_1}{t_1}}}{e^{-\frac{t-t_1}{t_1}} + 1} + \frac{1}{2(t_m + t)} e^{-\frac{(t-t_2)^2}{t_2^2}} - \frac{1}{t} \frac{e^{-\frac{t-t_3}{t_3}}}{e^{-\frac{t-t_3}{t_3}} + 1}. \quad (65)$$

The model contains a lot of free parameters, so we impose the following restriction on these: $t_s < t_1 < t_2 < t_3 < 0 < t_m$ and $l_1, l_2, l_3 > 0$ and in addition we assume that t is negative. The EoS for the Hubble rate of Eq. (65) is equal to,

$$w_{\text{eff}} = -1 - \frac{2 \left(-\frac{1}{\left(1 + e^{-\frac{t}{t_3}}\right)^2} \frac{1}{tt_3} + \frac{e^{-1+\frac{2t}{t_3}}(-t+t_3)}{(e+e^{t/t_3})t^2t_3} + \frac{e^{-\frac{(t-t_2)^2}{t_2^2}}(-t+t_2)}{t_2^2(1+t_m)} \right)}{3 \left(-\frac{e^{-1+\frac{2t}{t_3}}}{et + e^{t/t_3}t} + \frac{e^{-\frac{(t-t_2)^2}{t_2^2}}}{2(1+t_m)} + \frac{e(H_0+H_1(t-t_s)^\gamma)}{e+e^{t/t_1}} \right)^2} - \frac{2 \left(\frac{e^2(H_0+H_1(t-t_s)^\gamma)}{(e+e^{t/t_1})^2t_1} + \frac{e(H_0(-t+t_s)-H_1(t-t_s)^\gamma(t-t_s-t_1\gamma))}{(e+e^{t/t_1})t_1(t-t_s)} \right)}{3 \left(-\frac{e^{-1+\frac{2t}{t_3}}}{et+e^{t/t_3}t} + \frac{e^{-\frac{(t-t_2)^2}{t_2^2}}}{2(1+t_m)} + \frac{e(H_0+H_1(t-t_s)^\gamma)}{e+e^{t/t_1}} \right)^2}. \quad (66)$$

Then, when $t < t_1$, the first term dominates and a Type IV singularity occurs at $t = t_s$. Near the singularity, when $t \simeq t_s$, the Hubble rate is approximately equal to,

$$H(t) \simeq (H_0 + H_1 |t - t_s|^\gamma) \frac{e^{-\frac{t-t_1}{t_1}}}{e^{-\frac{t-t_1}{t_1}} + 1}, \quad (67)$$

while the effective EoS in this case is approximately equal to,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(-\frac{1}{\left(1 + e^{-\frac{t}{t_3}}\right)^2} \frac{1}{tt_3} + \frac{e^{-1+\frac{2t}{t_3}}(-t+t_3)}{(e+e^{t/t_3})t^2t_3} \right)}{3 \left(-\frac{e^{-1+\frac{2t}{t_3}}}{et+e^{t/t_3}t} + \frac{e(H_0+H_1(t-t_s)^\gamma)}{e+e^{t/t_1}} \right)^2} - \frac{2 \left(\frac{e^2(H_0+H_1(t-t_s)^\gamma)}{(e+e^{t/t_1})^2t_1} + \frac{e(H_0(-t+t_s)-H_1(t-t_s)^\gamma(t-t_s-t_1\gamma))}{(e+e^{t/t_1})t_1(t-t_s)} \right)}{3 \left(-\frac{e^{-1+\frac{2t}{t_3}}}{et+e^{t/t_3}t} + \frac{e(H_0+H_1(t-t_s)^\gamma)}{e+e^{t/t_1}} \right)^2}, \quad (68)$$

Consequently, near the Type IV singularity, the EoS can be further simplified to the following expression,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(-\frac{1}{\left(1 + e^{-\frac{t}{t_3}}\right)^2} \frac{1}{tt_3} + \frac{e^{-1+\frac{2t}{t_3}}(-t+t_3)}{(1+e^{t/t_3})t^2t_3} \right)}{3 \left(-\frac{e^{-1+\frac{2t}{t_3}}}{t+e^{t/t_3}t} + \frac{(H_0)}{e+e^{t/t_1}} \right)^2} - \frac{2 \left(\frac{e^2(H_0)}{(e+e^{t/t_1})^2t_1} \right)}{3 \left(-\frac{e^{-1+\frac{2t}{t_3}}}{t+e^{t/t_3}t} + e^{t/t_1} \right)^2}. \quad (69)$$

The EoS of Eq. (69) describes a quintessential acceleration if $t > t_3/2$, and phantom acceleration if otherwise. In the case that $t \sim t_2$, the second term dominates and a nearly de Sitter Universe can be realized, since the EoS in this case is approximately equal to,

$$w_{\text{eff}} \simeq -1 - \frac{8e^{-\frac{(t-t_2)^2}{t_2^2}}(-t+t_2)(1+t_m)}{3t_2^2}, \quad (70)$$

and since $t \simeq t_2$, we get $w_{\text{eff}} \simeq -1$. Furthermore when $t > t_3$, the last term dominates in Eq. (67) and the Universe becomes again a quintessential accelerating Universe, since the EoS in this case is,

$$w_{\text{eff}} \simeq \frac{1}{3} \left(-3 + 2e^{-\frac{4t}{t_3}} \left(et + e^{t/t_3}t \right)^2 \left(\frac{1}{tt_3} + \frac{e^{-1+\frac{2t}{t_3}}(t-t_3)}{(e+e^{t/t_3})t^2t_3} \right) \right). \quad (71)$$

As a final quite phenomenologically interesting model, we shall consider a modified variant singular form of the R^2 inflation model of Eq. (57), for which the Hubble rate reads,

$$H(t) = \frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma. \quad (72)$$

where δ and f_0 are arbitrary constant and positive parameters. In this case we assume that $\gamma, \delta > 1$, so that the cosmological evolution develops two Type IV singularities, one at $t = t_s$ and one at $t = t_0$. Also the cosmological time t_s is assumed to be at the end of inflation and the time instance t_0 is assumed to be much more later than t_s , so that $t_s \ll t_0$. Also if t_p represents the present time, then $t_0 \ll t_p$, so practically t_0 characterizes an intermediate cosmological era of evolution. In addition, the parameters H_0, H_i which are related to the Starobinsky model, are constrained by observational data to satisfy $H_0, H_i \gg 1$ (see [26]). Before we proceed, let us see which bigravity model can generate the cosmological evolution of Eq. (72), and the bigravity can be determined by calculating the functions $\omega(t)$, $V(t)$, $\sigma(t)$ and $U(t)$ appearing in Eq. (33). These functions for the Hubble rate of Eq. (33) read,

$$\begin{aligned} \omega(t) &= 2\sqrt{3} \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right), \\ \tilde{V}(t) &= -6 \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right)^2, \\ \sigma(t) &= \frac{12M_g^2 \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right)^2}{M_f^2}, \\ \tilde{U}(t) &= -6 \frac{M_g^2}{M_f^2} \left(m^2 M_{\text{eff}}^2 + 6 \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right)^2 \right) \\ &\quad \times \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right)^2, \end{aligned} \quad (73)$$

and again we took into account Eq. (24), so the above relations hold true near the singularity, where the conformal time is nearly identical to the cosmic time. Let us now proceed to the phenomenological implications of the model of Eq. (72). As we demonstrate, it has quite interesting phenomenological implications, since the early-time acceleration, the matter domination and also the late-time acceleration eras can be described by a single model. This is already obvious from the functional form of the Hubble rate, since at $t \simeq t_s$, the Hubble rate becomes approximately equal to,

$$H(t) \simeq H_0 + H_i(t-t_i), \quad (74)$$

since the term $\sim \frac{2}{3\left(\frac{4}{3H_0} + t\right)}$ is approximately equal to $\sim \frac{H_0}{2}$ and also the term $\sim (t-t_s)^\gamma$ is approximately equal to zero. So near $t = t_s$, which is assumed to occur near the early-time acceleration era, the model becomes nearly the Starobinsky R^2 inflation model, which is in concordance with observations [26, 32]. Let us now investigate what happens as the cosmic time evolves, and the best way to study the phenomenology is to study the EoS. The general form of the EoS for the Hubble rate (72), is equal to,

$$\begin{aligned} w_{\text{eff}} &= -1 - \frac{2 \left(e^{-(t-t_s)^\gamma} H_i - \frac{2}{3\left(\frac{4}{3H_0} + t\right)^2} + f_0(t-t_0)^\delta(t-t_s)^{-1+\gamma} \right)}{3 \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right)^2} \\ &\quad - \frac{2 \left(-e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) (t-t_s)^{-1+\gamma} + f_0(t-t_0)^{-1+\delta}(t-t_s)^\gamma \delta \right)}{3 \left(\frac{2}{3\left(\frac{4}{3H_0} + t\right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t-t_i) \right) + f_0(t-t_0)^\delta(t-t_s)^\gamma \right)^2}, \end{aligned} \quad (75)$$

so at early times and near $t \simeq t_s$, the EoS becomes,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(\frac{3H_0}{4} + H_i \right)}{3(H_0 + H_i(t - t_i))^2}, \quad (76)$$

and since a viable R^2 inflation model requires that H_0 and H_i to be very large [26], the EoS is approximately equal to $w_{\text{eff}} \simeq -1$, which is a de Sitter accelerating phase, as was expected. At $t \simeq t_0$, which is much after the time instance t_s , the EoS becomes approximately equal to,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(e^{-(t-t_s)^\gamma} H_i - \frac{2}{3 \left(\frac{4}{3H_0} + t \right)^2} - e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t - t_i) \right) (t - t_s)^{-1+\gamma\gamma} \right)}{3 \left(\frac{2}{3 \left(\frac{4}{3H_0} + t \right)} + e^{-(t-t_s)^\gamma} \left(\frac{H_0}{2} + H_i(t - t_i) \right) \right)^2}. \quad (77)$$

It is obvious that since $t \sim t_0 \gg t_s$, and also $t \gg \frac{4}{3H_0}$ the above expression becomes,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(e^{-t^\gamma} H_i - \frac{2}{3t^2} - e^{-t^\gamma} \left(\frac{H_0}{2} + H_i(t - t_i) \right) t^{-1+\gamma\gamma} \right)}{3 \left(\frac{2}{3t} + e^{-t^\gamma} \left(\frac{H_0}{2} + H_i(t - t_i) \right) \right)^2}, \quad (78)$$

and since the $t \gg 1$, the terms proportional to $\sim e^{-t^\gamma}$ are exponentially suppressed. In effect, the EoS becomes,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(-\frac{2}{3t^2} \right)}{3 \left(\frac{2}{3t} \right)^2}, \quad (79)$$

which is approximately equal to zero. So the era near $t \simeq t_0$ describes a matter domination era in this case since $w_{\text{eff}} \simeq 0$. Finally, at $t \simeq t_p$, the EoS becomes approximately equal to,

$$w_{\text{eff}} \simeq -1 - \frac{2 \left(e^{-t^\gamma} H_i - \frac{2}{3t^2} + f_0 t^{-1+\gamma+\delta} \gamma - e^{-t^\gamma} t^{-1+\gamma} \left(\frac{H_0}{2} + H_i(t - t_i) \right) \gamma + f_0 t^{-1+\gamma+\delta} \delta \right)}{3 \left(\frac{2}{3t} + f_0 t^{\gamma+\delta} + e^{-t^\gamma} \left(\frac{H_0}{2} + H_i(t - t_i) \right) \right)^2}, \quad (80)$$

so by omitting the exponentially suppressed terms and also since the term $\sim \frac{1}{t}$ is subdominant at times $t \sim t_p$, compared to the positive powers of t , the EoS becomes finally,

$$w_{\text{eff}} \simeq -1 - \frac{2t^{-1-\gamma-\delta}\gamma}{3f_0} - \frac{2t^{-1-\gamma-\delta}\delta}{3f_0}. \quad (81)$$

Since $t \sim t_p$ and t_p is approximately of the order $t \sim 10^{17}$ sec, this means that the terms $\sim t^{-1-\gamma-\delta}$ satisfy $t^{-1-\gamma-\delta} \ll 1$, owing to the fact that we initially assumed $\gamma, \delta > 1$, so that two Type IV singularities occur. Therefore, at late-time we have $w_{\text{eff}} \simeq -1$, and hence, the Universe is described by a nearly de Sitter evolution, slightly crossing the phantom divide. In fact, such an evolution is supported by present time observations which predict that $w_{\text{eff}} \simeq -1$, but with the EoS slightly crossing the phantom divide [35]. Hence, with the model of (72) we were able to describe within the same theoretical framework, three cosmological eras, an early de Sitter acceleration era, a matter domination era, and a late-time acceleration era. In Table II we present the behavior of the EoS for the modified singular R^2 inflation model of Eq. (72) for the various cosmological eras.

TABLE II: Behavior of Equation of State for the Modified Singular R^2 Inflation Model of Eq. (72)

Cosmological Time	EoS w_{eff}	Evolution Type
$t \simeq t_s$	$w_{\text{eff}} \simeq -1$	Nearly de Sitter
$t \simeq t_0$	$w_{\text{eff}} \simeq 0$	Matter Domination
$t \simeq t_p$	$w_{\text{eff}} \simeq -1$	Nearly de Sitter

VI. CONCLUSIONS

In this paper we demonstrated how a Type IV singular evolution can be successfully generated by a bimetric $F(R)$ gravity theory. Particularly, by employing the formalism of bimetric $F(R)$ gravity theory, we were able to describe two singular evolutions, one of which is a singular variant of the Starobinsky R^2 inflation model. By calculating the slow-roll parameters ϵ , η and ξ , we showed that in some cases these parameters contain singularities. Hence, one could claim that the observable quantities are singular and therefore the singularities lead to unphysical results, but this is not the case, since the singularities in the slow-roll parameters indicate that a strong instabilities occur at the time instance that the singularity occurs. Indeed, the slow-roll parameters are the lowest order terms in the slow-roll expansion, and hence an abrupt singular increase of their values indicates that the slow-roll expansion breaks down. Therefore, the cosmological dynamical system becomes unstable at the singularity point and therefore the solution that described the inflationary solution up to that point, ceases to be the final attractor of the theory and therefore a new attractor is chosen by the theory. Hence, the singularities in the slow-roll parameters show that graceful exit is triggered at that point. Of course the presence of singularities per se is not sufficient for proving that inflation indeed ends at the singularity, but another underlying mechanism probably controls the exit, like curvature perturbations around unstable de Sitter vacua, as it was the case in [26, 33], or a tachyonic instability exists in the theory.

Apart from the early-time behavior, we examined the late-time behavior of the models we mentioned earlier, and we showed that the unified description of early and late-time acceleration was possible in the context of bimetric $F(R)$ gravity. In addition, in one of the models we studied, we showed that three different eras can be successfully described by using a single model, namely, early time acceleration, the matter domination era, and also the late-time acceleration era.

What would be of fundamental importance is to find a mechanism to describe the graceful exit from inflation, since in the present study we provided some sufficient proof that graceful exit might occur at the time instance that the singularity occurs, but we did not prove that graceful exit indeed occurs. It is therefore important to investigate in the Jordan frame, with which mechanism the graceful exit can actually occur. In addition, we should also investigate if the second metric, the fiducial one, can be distinguished phenomenologically from the physical metric, at the level of cosmological perturbations. In this paper we took into account a flat Minkowski fiducial metric, but this is the simplest case one can choose. In principle, one could also choose a non-flat FRW metric, so this could perplex the study to a great extent. This question will be considered elsewhere.

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Appendix: Detailed Form of Some Intermediate Expressions

In this Appendix we present the detailed form of the slow-roll indices corresponding to the singular R^2 model with Hubble rate appearing in Eq. (57). We start off with the parameter ϵ , with its full form being equal to,

$$\begin{aligned} \epsilon \simeq & -3 \left(\frac{(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^\gamma)^2 (-6H_i + M^2(-1+t-t_i) - 6f_0(t-t_s)^{-1+\gamma}(t-t_s-\gamma))}{Q(t)} \right. \\ & + \frac{+\frac{1}{36}(6H_i - M^2(-1+t-t_i) + 6f_0(t-t_s)^{-1+\gamma}(t-t_s-\gamma))^2 + \frac{M^2(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^\gamma)^2}{3}}{Q(t)} \\ & + \frac{(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^\gamma)^4 - 2f_0(t-t_s)^{-1+\gamma}\gamma(H_i - \frac{1}{6}M^2(t-t_i) + f_0(t-t_s)^\gamma)^2}{Q(t)} \\ & \left. + \frac{f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma - \left(-\frac{M^2}{6} + f_0(t-t_s)^{-1+\gamma}\gamma\right)^2}{Q(t)} \right), \end{aligned} \quad (82)$$

where the function $Q(t)$ stands for,

$$Q(t) = 2 \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right)^5 \left(3 + \frac{-6H_i + M^2(-1 + t - t_i) - 6f_0(t - t_s)^{-1+\gamma}(t - t_s - \gamma)}{6 \left(H_i - \frac{1}{6} M^2 (t - t_i) + f_0 (t - t_s)^\gamma \right)^3} \right)^2 \times (-6H_i + M^2(-1 + t - t_i) - 6f_0(t - t_s)^{-1+\gamma}(t - t_s - \gamma)). \quad (83)$$

Correspondingly, the full form of the slow-roll parameter η is,

$$\begin{aligned} \eta \simeq & -\frac{1}{4 \left(3 - \frac{36(6H_i + M^2(1-t+t_i))}{(6H_i + M^2(-t+t_i))^3} \right)} \\ & \times \left(-\frac{648(6H_i + M^2(1-t+t_i))}{(6H_i + M^2(-t+t_i))^3} + \frac{36(6H_i + M^2(1-t+t_i))^2}{(6H_i + M^2(-t+t_i))^4} \right. \\ & + \frac{36 \left(\frac{M^2}{3} + (H_i + \frac{1}{6} M^2(-t+t_i))^2 - \frac{M^4}{(6H_i + M^2(-t+t_i))^2} + \frac{f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(H_i + \frac{1}{6} M^2(-t+t_i))^2} \right)}{-6H_i + M^2(-1+t-t_i)} \\ & + \frac{3 \left(\frac{M^2}{3} + (H_i + \frac{1}{6} M^2(-t+t_i))^2 - \frac{M^4}{(6H_i + M^2(-t+t_i))^2} + \frac{f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(H_i + \frac{1}{6} M^2(-t+t_i))^2} \right)}{(H_i + \frac{1}{6} M^2(-t+t_i))^2} \\ & - \frac{\left(\frac{M^2}{3} + (H_i + \frac{1}{6} M^2(-t+t_i))^2 - \frac{M^4}{(6H_i + M^2(-t+t_i))^2} + \frac{f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(H_i + \frac{1}{6} M^2(-t+t_i))^2} \right)^2}{(H_i + \frac{1}{6} M^2(-t+t_i))^4} \\ & + \frac{12 \left(-H_i^2 + \frac{1}{3} H_i M^2(t - t_i) + \frac{1}{36} M^4 \left(-t^2 + 2tt_i - t_i^2 - \frac{36}{(6H_i + M^2(-t+t_i))^2} \right) \right)}{-6H_i + M^2(-1+t-t_i)} \\ & \left. + \frac{12 \left(-\frac{36f_0(t-t_s)^{-3+\gamma}(2+3t-3t_s-\gamma)(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^2} + M^2 \left(-\frac{5}{6} + \frac{72f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^3} \right) \right)}{-6H_i + M^2(-1+t-t_i)} \right). \quad (84) \end{aligned}$$

Finally, the full form of slow-roll parameter ξ is too lengthy to quote here, so we give only the terms that contain singularities, so the singular part of ξ , which we denote as ξ_s , is equal to,

$$\begin{aligned} \xi_s = & \frac{f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{4 \left(H_i + \frac{1}{6} M^2(-t+t_i) \right)^4 \left(3 - \frac{36(6H_i + M^2(1-t+t_i))}{(6H_i + M^2(-t+t_i))^3} \right)} \\ & \times \left(\frac{4f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(H_i + \frac{1}{6} M^2(-t+t_i))^4} + \frac{90f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(-6H_i + M^2(-1+t-t_i))(H_i + \frac{1}{6} M^2(-t+t_i))^2} \right. \\ & - \frac{648f_0^2(t-t_s)^{-4+2\gamma}(-1+\gamma)^2\gamma^2}{(-6H_i + M^2(-1+t-t_i))^3(H_i + \frac{1}{6} M^2(-t+t_i))^2} - \frac{36f_0^2(t-t_s)^{-4+2\gamma}(-1+\gamma)^2\gamma^2}{(H_i + \frac{1}{6} M^2(-t+t_i))^4(6H_i + M^2(1-t+t_i))^2} \\ & + \frac{1296f_0^3(t-t_s)^{-6+3\gamma}(-1+\gamma)^3\gamma^3}{(H_i + \frac{1}{6} M^2(-t+t_i))^4(6H_i + M^2(1-t+t_i))^4} \\ & + \frac{30 \left(\frac{72f_0M^2(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^3} - \frac{36f_0(t-t_s)^{-3+\gamma}(2+3t-3t_s-\gamma)(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^2} \right)}{-6H_i + M^2(-1+t-t_i)} \\ & + \frac{3M^4 \left(H_i + \frac{1}{6} M^2(-t+t_i) \right)^2 \left(\frac{72f_0M^2(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^3} - \frac{36f_0(t-t_s)^{-3+\gamma}(2+3t-3t_s-\gamma)(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^2} \right)}{(6H_i + M^2(1-t+t_i))^2} \\ & - \frac{432f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma \left(\frac{72f_0M^2(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^3} - \frac{36f_0(t-t_s)^{-3+\gamma}(2+3t-3t_s-\gamma)(-1+\gamma)\gamma}{(6H_i + M^2(-t+t_i))^2} \right)}{(-6H_i + M^2(-1+t-t_i))^3} \\ & + \frac{36 \left(H_i + \frac{1}{6} M^2(-t+t_i) \right)^2}{(6H_i + M^2(1-t+t_i))^2} \left(-\frac{4f_0(t-t_s)^{-3+\gamma}(-2+\gamma)(-1+\gamma)\gamma}{(H_i + \frac{1}{6} M^2(-t+t_i))^2} \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{f_0(t-t_s)^{-4+\gamma}(-3+\gamma)(-2+\gamma)(-1+\gamma)\gamma}{\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^2} \Bigg) + \frac{f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{4\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^4 \left(3 - \frac{36(6H_i+M^2(1-t+t_i))}{(6H_i+M^2(-t+t_i))^3}\right)} \\
& \times \left(\frac{4f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^4} + \frac{90f_0(t-t_s)^{-2+\gamma}(-1+\gamma)\gamma}{(-6H_i + M^2(-1+t-t_i))\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^2} \right. \\
& \left. - \frac{648f_0^2(t-t_s)^{-4+2\gamma}(-1+\gamma)^2\gamma^2}{(-6H_i + M^2(-1+t-t_i))^3\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^2} - \frac{36f_0^2(t-t_s)^{-4+2\gamma}(-1+\gamma)^2\gamma^2}{\left(H_i + \frac{1}{6}M^2(-t+t_i)\right)^4(6H_i + M^2(1-t+t_i))^2} \right) \Bigg) .
\end{aligned} \tag{85}$$

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